* Weddle's rule

Putting $n=6$ in equation ①, Newton-cotes formula

$$\int_{x_0}^{x_0+6h} f(x)\,dx = h\Big[6y_0 + 18\Delta y_0 + \frac{1}{2}(72-18)\Delta^2 y_0 + \frac{1}{6}(324-216+36)\Delta^3 y_0 + \cdots\Big]$$

$$= h\Big[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\Big]$$

Replace the term $\frac{41}{140}\Delta^6 y_0$ by $\frac{42}{140}\Delta^6 y_0$. By this change, the error introduced is only $\frac{h}{140}\Delta^6 y_0$, which is negligible when h and $\Delta^6 y_0$ are small. Using $\Delta = E-1$ and replacing all differences in terms of y's, we get

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

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$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}\left[y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}\right]$$

$\vdots$

$$\int_{x_0+(n-6)h}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\left[y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n\right]$$

Adding all these integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}\Big[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5) + (2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11}) + (\cdots) + (2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)\Big] \rightarrow (3)$$

Equation (3) is called Weddle's rule.

Problems:

1). Evaluate $\int_{-3}^{3} x^4 dx$ by using (i) Trapezoidal rule (ii) Simpson's rule. Verify your results by actual integration.

Sol:

Here $y(x) = x^4$. Interval length $(b-a) = 6$. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$. We form below the table.

x:	-3	-2	-1	0	1	2	3
y:	81	16	1	0	1	16	81

(i) By Trapezoidal rule,

$$\int_{-3}^{3} y\, dx = \frac{h}{2}[\text{sum of the first and last ordinates} + 2(\text{sum of the remaining ordinates})].$$

$$= \frac{1}{2}\left[(81+81) + 2(16+1+0+1+16)\right]$$

$$= 115.$$

(ii) By Simpson's one-third rule (since number of ordinates is odd).

$$\int_{-3}^{3} y\,dx = \frac{1}{3}\left[(81+81)+2(1+1)+4(16+0+16)\right]$$

$$= 98.$$

(iii) Since $n=6$, (multiple of three), we can also use simpson's three-eights rule. By this rule,

$$\int_{-3}^{3} y\,dx = \frac{3}{8}\left[(81+81)+3(16+1+1+16)+2(0)\right]$$

$$= 99.$$

(iv) By actual integrations,

$$\int_{-3}^{3} x^4\,dx = 2\left(\frac{x^5}{5}\right)_0^3$$

$$= \frac{2\times 243}{5}$$

$$= 97.2$$

From the results obtained by various methods, we see that Simpson's rule gives better result than Trapezoidal rule.

2) Evaluate the integral $I = \int_4^{5.2} \log_e x \, dx$ using Trapezoidal, Simpson's and Weddle's rule.

Sol

Here $b-a = 5.2-4 = 1.2$. We shall divide the interval into 6 equal parts.

Hence, $h = \frac{1.2}{6} = 0.2$. We form the table.

x	: 4	4.2	4.4	4.6	4.8
$f(x) = \log_e x$	: 1.3862944	1.4350845	1.4816045	1.5260563	1.5686159

x	: 5.0	5.2
$f(x)$	: 1.6094379	1.6486586.

Sol

(i) By Trapezoidal rule,

$$\int_4^{5.2} \log x \, dx = \frac{0.2}{2}\Big[(1.3862944 + 1.6486586 + 2(1.4350845 + 1.4816045 + 1.5260563 + 1.5686159 + 1.6094379)\Big]$$

$$= 1.82765512.$$

(ii) Since $n=6$, we can use Simpson's rule and Weddle's rule also.

By Simpson's one third rule,

$$I = \frac{0.2}{3}\left[(1.3862944 + 1.6486586) + 2(1.4816045 + 1.5686159) + 4(1.4350845 + 1.5260563)\right]$$

$$= 1.82784724$$

(iii) By Simpson's three-eights rule,

$$I = \frac{3(0.2)}{8}\left[(1.3862944 + 1.6486586) + 3(1.4350845 + 1.4816045 + 1.5686159 + 1.6094379 + 2(1.5260563)\right]$$

$$= 1.82784705.$$

(iv) By Weddle's rule,

$$I = \frac{3(0.2)}{8}\left[1.3862944 + 5(1.4350845) + 1.4816045 + 6(1.5260563) + 1.5686159 + 5(1.6094379) + 1.6486586\right]$$

$$= 1.82784739.$$